Santa Claus and Conservation of Energy

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Abstract

This is a pedagogical paper on an amusing application of the concept of the kinetic energy (KE). Using some rudimentary physical notions, we have analyzed the energetics of the motion of Santa Claus. The results, which are quite surprising, can be of interest to high school and early college physics educators when they teach KE, and energy conservation in general.

Energy and its conservation is arguably one of the most fundamental concepts in physics, and its teaching requires careful planning and effective pedagogy. Once the formalism is introduced, introducing students to as many examples as possible is extremely helpful. Routine examples abound in many textbooks, but interesting examples, the ones that *really* get the attention of students are not as easy to come by. The analysis of the speed and especially the energy consumption of Santa Claus can be a delightful experience for both the teacher and students of introductory physics.

1 Santa Claus's Energy Usage: Formalism

Santa Claus's consumption of energy and heat production is the focus of this section. Because this energy turns out to be enormous, we have to consider a motion that minimizes this consumption. With N_c the number of Christian children and n the number of children per home, the number of chimneys Santa has to visit is N_c/n, and the time (call it T) available to him for each chimney is $24 \times 3600/(N_c/n)$ or $T = 86400n/N_c$ second.

Santa cannot afford to spend too much time in each house, because he has only 24 hours to deliver all the toys he is carrying. On the other hand, he cannot spend too little time moving through a chimney, because then he may need too much fuel for the resulting huge kinetic energy. We need to calculate the optimal time spent per chimney, i.e., the time that minimizes Santa's energy consumption. Let t stand for the time of flight down (or up) a chimney. We want to write Santa's energy consumption in terms of t and find the t that minimizes the energy. We assume that the only energy Santa uses is the energy required to speed up (instantaneously) to his final speed. There are thus only two kinds of usage: for traveling down and up the chimneys and for hopping from one chimney to the next. Denote the first one by KE_{chim} and the second one by KE_{hop}.

With M and m denoting Santa's mass and the mass of each toy, respectively, we can write

$$KE_{\text{chim}} = \underbrace{\tfrac{1}{2}(nm + M)\left(\frac{h}{t}\right)^2}_{\text{going down}} + \underbrace{\tfrac{1}{2}M\left(\frac{h}{t}\right)^2}_{\text{coming up}} = \tfrac{1}{2}(nm + 2M)\left(\frac{h}{t}\right)^2 \tag{1}$$

where h is the height of the chimney.

The hopping energy is a little more complicated. Santa starts out with a lot of toys; but as he delivers the toys, their number decreases. Intuitively, we may want to take the *effective* mass of santa plus the toys to be half the mass he starts with (Santa's mass is negligible compared with the total mass of the toys). It turns out that an exact calculation of the hopping kinetic energy leads to the same result (see the Appendix). Therefore,

$$KE_{\text{hop}} = \tfrac{1}{2}\left(\frac{N_c m}{2}\right) v^2 = \left(\frac{N_c m}{4}\right)\left(\frac{d}{T-2t}\right)^2 \tag{2}$$

where d is the (average) distance between two consecutive houses and T is as defined before. Thus, as a function of t, the total energy per house is

$$KE(t) = KE_{\text{hop}} + KE_{\text{chim}} = \left(\frac{N_c m}{4}\right)\left(\frac{d}{T-2t}\right)^2 + \tfrac{1}{2}(nm+2M)\frac{h^2}{t^2}, \tag{3}$$

Differentiating Equation (3) with respect to t and setting the derivative equal to zero, we obtain

$$\frac{N_c m d^2}{(T-2t)^3} = \frac{h^2(nm+2M)}{t^3} \quad \text{or} \quad \frac{T-2t}{t} = \left(\frac{N_c m d^2}{h^2(nm+2M)}\right)^{1/3}$$

This yields

$$t = \frac{T}{a+2}, \qquad a \equiv \left(\frac{N_c m d^2}{h^2(nm+2M)}\right)^{1/3}, \qquad T \equiv \frac{86400n}{N_c} \tag{4}$$

If the notion of derivative is unfamiliar to students, one can substitute some reasonable numbers (as done in the next section) in Equation (3), plot $KE(t)$, and read off the t corresponding to the minimum of $KE(t)$.

2 Santa Claus's Energy Usage: Numerics

We now put some reasonable numbers in the foregoing equations to get a feel for the magnitude of the quantities associated with Santa's motion.

Let us start by estimating the number of children and the number of houses. There are approximately 1.5 billion Christians in the world, and—estimating drastically in Santa's favor—let us assume that eligible children constitute only 10 percent of this population. Thus, $N_c = 1.5 \times 10^8$. To make life really easy for Santa, let us assume a (very high) concentration of 10 children per house. This brings the number of houses down to 1.5×10^7, and makes $T = 0.00576$ s.

To further help Santa in his seemingly impossible task, we bring the houses next to each other, make them really small, place them side by side on a straight line so that the distance between consecutive chimneys is only 10 meters, for each we construct a short chimney of only 4 meters, make Santa a thin 100-kg person, and assume a light average mass of 2 kg per toy. These parameters yield a value of 204.26 for a, and a numerical formula

$$KE(t) = \frac{7.5 \times 10^9}{(0.00576 - 2t)^2} + \frac{1760}{t^2}$$

which can be plotted as a funtion of t. Using the numerical value of a in (4), or reading off from the plot of $KE(t)$, the t that minimizes the energy is found to be 0.0000279 s, or 0.0000558 seconds per chimney; i.e., Santa will have to cover about 18000 chimneys every second! Since this does not violate any physical laws, let us accept it.

What is more significant is Santa's energy consumption, which we have assumed to be used simply to speed him (and his toys) up. The energy corresponding to the motion down a chimney is

$$KE_{\text{down}} = \tfrac{1}{2}(nm + M)\left(\frac{h}{t}\right)^2 = \tfrac{1}{2}(120)\left(\frac{4}{0.0000279}\right)^2 = 1.23 \times 10^{12} \text{ Joules},$$

with a slightly lower energy (1.03×10^{12} J, because he leaves the toys in the house) for the motion up the chimney.

2.1 A Comparison

The "hopping" energy is

$$KE_{\text{hop}} = \left(\frac{(1.5 \times 10^8)(2)}{4}\right)\left(\frac{10}{0.00576 - 0.0000558}\right)^2 = 2.3 \times 10^{14} \text{ J}$$

which is about 100 times larger than KE_{chim}; so in calculating the total energy, we can ignore the latter. For all the 15 million houses, Santa needs 3.3×10^{21} J. This is a huge amount of energy. Nevertheless, it is the minimum amount Santa can spend. But what does it really mean? To get a feel for its magnitude, we have to compare it with another huge quantity of energy. For example, how does it compare with the yearly energy consumption of the world? The entire annual energy consumption of the world was about 4×10^{20} Joules in 2001 [1]. This includes not only the typical residential usage such as heating, lighting, cooking, commuting and entertainment, but also the large scale industrial, agricultural, and transportation consumption. In short, it is the entire energy used in the world in all its shapes and forms. Santa uses over 8 times the annual world energy supply *in one day*! Thus, the entire world must stop using any form of energy for over 8 years so that Santa will be able to deliver his toys in one day. Clearly Noel's visit cannot be an annual event. At best it can be done every decade, for the preparation of which the whole population of the world (Christian and non-Christian) must stop consuming any form of energy!

2.2 The Explosions

Although extremely difficult, the people of the world might be willing to tolerate all the harshness caused by Santa's trip were it not for the revelation that behind his jolly "ho, ho, ho" there is destruction. Huge destruction! Far beyond the kind of destruction that one hears about in terrorists' bomb attacks! Of course, Santa's destructions are not intentional, but he just can't help *exploding* houses as he visits them! How can that be?

An explosion is simply the release of a large amount of energy in a small time period. Take Santa's plunge down a typical chimney. We found that he has a kinetic energy of 1.23×10^{12} Joules when he reaches the bottom of the chimney. This energy turns into heat when he brakes to a complete stop. For comparison, the heat produced in the explosion of a ton of TNT is about 4×10^9 Joules. Thus, Santa releases the equivalent of 307.5 tons of TNT when landing in the house, and almost the same amount when he climbs up the chimney and stops at the roof! And a time interval of less than 27.9 microseconds for each of these releases is short enough to qualify them as explosions.

But the real killer is the explosion caused by his landing at the chimney as he comes from the previous house. His KE as he lands at the chimney is 2.3×10^{14} Joules, and this energy is turned into heat in less than 0.00576 second, qualifying this release of energy as an explosion as well. How many tons of TNT is this equivalent to? $2.3 \times 10^{14}/4 \times 10^9 = 57500$ tons of TNT! Suffice it to say that the destructive power of "Little Boy," the bomb that was dropped on Hiroshima, was a "mere" 15000 tons of TNT. Every time Santa lands at a chimney, he detonates about 4 Hiroshima-type bombs, and he is at ground zero of every blast! Santa is the ultimate "suicide bomber," who survives the equivalent of 15 million gigantic nuclear blasts every year!

A Appendix

Recalling that N_c/n is the number of houses, the total hopping KE can be written as

$$\begin{aligned}
KE =& \tfrac{1}{2}(N_c m + M)v^2 + \tfrac{1}{2}[(N_c - n)m + M]v^2 + \tfrac{1}{2}[(N_c - 2n)m + M]v^2 + \cdots \\
=& \frac{N_c}{n}\left[\tfrac{1}{2}(N_c m + M)v^2\right] - \tfrac{1}{2}mv^2 n\left(1 + 2 + 3 + \cdots + \frac{N_c}{n} - 1\right) \\
=& \frac{N_c}{n}\left[\tfrac{1}{2}(N_c m + M)v^2\right] - \tfrac{1}{2}mv^2 n\left[\frac{N_c/n(N_c/n - 1)}{2}\right] \\
=& \tfrac{1}{4}\frac{N_c^2}{n}mv^2 + \tfrac{1}{2}\left(\frac{N_c}{n}M + \frac{mN_c}{2}\right)v^2
\end{aligned}$$

Dividing this by the number of houses, we get

$$KE_{\text{hop}} = \tfrac{1}{4}N_c mv^2 + \tfrac{1}{2}\left(M + \frac{mn}{2}\right)v^2 \approx \tfrac{1}{2}\left(\frac{N_c m}{2}\right)v^2$$

because $N_c m$ is much greater than M and $mn/2$.

References

[1] See http://www.eia.doe.gov/oiaf/ieo/world.html, the US Department of Energy web site.