A heuristic derivation of Minkowski distance and Lorentz transformation

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Abstract

Students learn new abstract concepts best when these concepts are connected through a well-designed analogy, to familiar ideas. Since the concept of the relativistic spacetime distance is too abstract, it would be desirable to connect it to the familiar Euclidean distance, but present the latter in such a way that it makes transparent contact with the former. Starting with some intuitive and "obvious" assumptions concerning distance in one dimension, we "derive" the two-dimensional Euclidean distance between two points in terms of their coordinates. Then, assuming the invariance of this distance, we derive the (familiar) two-dimensional orthogonal coordinate transformation. We present the derivation in such a way that the transition to spacetime becomes "self-evident." Thus, following exactly the same procedure, we derive the Minkowskian distance and the corresponding transformation that respects the invariance of that distance, i.e., the Lorentz transformation.

1 Introduction

The teaching of relativity to beginning students can be a very challenging task for the teacher, and the counterintuitive concepts of length contraction, time dilation, massless photon, etc., can be a daunting experience for the student. Although one can easily derive time dilation formula based entirely on the simple postulate of relativity, namely *the constancy of the speed of light* [see the discussion in Section 3 leading up to Equation (6)] and use it to get to the length contraction formula, the rest of the relativity theory will be out of reach until and unless the Minkowski distance and Lorentz transformation are introduced. Therefore, the challenge of teaching the theory rests on these two important concepts.

Some textbooks use the postulational approach [1] whereby one accepts the Minkowski distance and only studies its consequences. Other textbooks rely heavily on various "diagrams" [2] obeying rules whose origin is not elucidated. A third category of textbooks use H. Bondi's k-calculus [3] to arrive at the Lorentz transformation, and "discover" that such a transformation leaves the Minkowski distance invariant. All these approaches seem ad hoc and do not "convince" the first-timers that *the metric is truly based on experimental observations*. Thus, although many of the students will go on to use the metric to solve problems, the question of where the Minkowski distance and Lorentz transformation came from remains unanswered.

In this paper, we try to present some "convincing" argument that Minkowski distance and Lorentz transformation are the expected consequences of the postulates of relativity, which themselves are based entirely on observations and common sense. We start with the"derivation" of the familiar Euclidean (Pythagorean) distance based on some very intuitive and obvious assumptions. We then use the invariance of this distance to find out how different coordinate systems are related to each other. Having the experience of this familiar case behind us, we employ the ideas used in the Euclidean case to motivate the Minkowski distance of special relativity in a very natural way.

2 "Derivation" of Euclidean distance

Flatlanders live in a three-dimensional world, but are aware of only two dimensions called *length* and *width*. Although they do not have a physical experience with the third dimension, *height*, they have ways of measuring it with an *acrometer*. Carrying an acrometer, each flatlander can tell how "high" (s)he has "climbed" from some initial point.

Flatlanders know how to describe a point in terms of a coordinate system (they choose x and y for the two dimensions in which they live) drawn in their flat land and how to express the distance between them in terms of their coordinates: if points P_1 and P_2 have coordinates (x_1, y_1) and (x_2, y_2), respectively, then their distance is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

The flatlanders' "elevators" have a strange but important property. Generally, an elevator in vertical motion is also seen to be moving horizontally. However, for some observers, a particular set of elevators, which we call *proper elevators*, do not move in the length and width directions, although the elevators are known to be moving because the riders inside can see the dials on their acrometer change. These same (moving elevators) move in the flatland of other observers. We say that the first group of observers belong to the same *reference frame* (RF) and the proper elevators designate that RF. Each RF has its own proper elevators, and these elevators distinguish among different RFs. In fact, we can quantify the distinction, and define the relative *slope* of two RFs as follows.

Let Bob take one of his proper elevators while Alice looks at the motion of this elevator in her RF. She keeps track of both the position (x, y) and height z of the elevator as it moves in her two-dimensional world. Suppose that the elevator is found at point P_1 with coordinates (x_1, y_1) and at height z_1, and a little later at point P_2 with coordinates (x_2, y_2) and at height z_2. Then the slope, which is a vector quantity, is defined as

$$\mathbf{m} \equiv \frac{(x_2 - x_1, y_2 - y_1)}{z_2 - z_1} \equiv \frac{(\Delta x, \Delta y)}{\Delta z} = \left(\frac{\Delta x}{\Delta z}, \frac{\Delta y}{\Delta z}\right)$$

We consider only "constat-realtive-slope" observers. This means that $\mathbf{m}$ depends only on Δx, Δy, and Δz, and not on where Alice started her observation. So, if the elevator moves from P_1' to P_2' and the difference between the coordinates and heights of these two points are $\Delta x'$, $\Delta y'$, and $\Delta z'$, then the slope $\mathbf{m} = (\Delta x', \Delta y')/\Delta z'$ is the same as before.

Since height is a property that seems to be present in all aspects of the flatlanders' physical universe, and it is a measurable quantity, they find it convenient to treat height as a "third dimension" and define *events* as points of this mysterious three-dimensional space. Thus, an event E is described by three numbers (x, y, z). Furthermore, they want to generalize their distance formula (1) to include this new dimension. Let's ignore the y-coordinate, because we know how to incorporate it in the distance formula, and concentrate on the new two-dimensional geometry of the xz-plane. So, the big challenge is to find a distance formula for two events having coordinates (x_1, z_1) and (x_2, z_2).

The generalization can be dictated only by observation. As our experience with Euclidean geometry has shown, a cherished and "obvious" assumption such as the Euclid's fifth postulate can lead only to one geometry out of many possible geometries. So, the question of which geometry is the right geometry can be answered only by observation.

To describe this observation, let's go back to Bob and Alice and have them perform a crucial experiment. Bob, standing at the origin of his coordinate system and riding his proper elevator, is to record the heigh when his elevator passes the point P_1 of Alice's RF and then again at P_2, while Alice records the heights and x-coordinats of the points. Let Z_1 and Z_2 be the heights as measured by Bob, and (x_1, z_1) and (x_2, z_2), the coordinates measured by Alice. The result of this important experiment can be summarized as

$$z_2 - z_1 = \frac{Z_2 - Z_1}{\sqrt{1 + m^2}} \tag{2}$$

where m is the slope of Bob's RF relative to Alice's. Equation (2) is a very important relation, because as we shall see below, it alone determines the form the generalized distance takes.

If (the generalization of the) distance is to have any meaning, we have to assign to it the following obvious property: *If two events happen to have equal coordinates in all but one dimension, then the distance is the difference between the values of the unequal coordinates.* Thus, for the two events E_1 and E_2 above, whose coordinates according to Bob are $(0, Z_1)$ and $(0, Z_2)$, the distance *must* be

$$d(E_1, E_2) = Z_2 - Z_1$$

assuming that $Z_2 > Z_1$. Now noting that $m = (x_2 - x_1)/(z_2 - z_1)$, we obtain

$$\begin{aligned} d(E_1, E_2) &= Z_2 - Z_1 = (z_2 - z_1)\sqrt{1 + m^2} = (z_2 - z_1)\sqrt{1 + ((x_2 - x_1)/(z_2 - z_1))^2} \\ &= \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \end{aligned} \tag{3}$$

This is of course the familiar Euclidean distance, which we, the creatures familiar with the third dimension have known for a long time. However, precisely because of this familiarity, we can appreciate the ingenuity of the flatlanders in arriving at the same result without having a physical contact with the third dimension. *It is to be emphasized that the distance formula (3) depends crucially on the purely empirical result in Equation (2).*

Equation (3), being the distance between two events, must not depend on the coordinates used to describe them. What kind of coordinate transformation leaves (3) unchanged? Appendix A shows that if E has coordinates (x, z) relative to observer O and (x', z') relative to observer O', then

$$\begin{aligned} x' &= a + \frac{1}{\sqrt{1 + m^2}}x + \frac{m}{\sqrt{1 + m^2}}z = a + x\cos\alpha + z\sin\alpha \\ z' &= b - \frac{m}{\sqrt{1 + m^2}}x + \frac{1}{\sqrt{1 + m^2}}z = b - x\sin\alpha + z\cos\alpha \end{aligned} \tag{4}$$

where m is the slope of O' relative to O and α is the angle between the axes of O and O'.

Figure 2 sheds some light on the mysterious properties of "height" and the strange behavior of elevators: Bob's proper elevator moves along the z-axis, and thus has no projection on the x-axis. Therefore, it does not move in the O reference frame. However, the same elevator moves in the x' direction. Furthermore, the slope determined by the motion of elevators is precisely the slope with which we are familiar.

3 Minkowskian distance

We are three-dimensional creatures living in a four-dimensional *spacetime*, and having no physical contact with the fourth dimension. However, just like the flatlanders, we have instruments, our clocks, which can measure the invisible fourth dimension. We can tell if something occurred before or after something else, but cannot picture these the same way we do two buildings that are 500 meters apart.

Just like the flatlanders, we want to find a formula for the distance between two events (points in the spacetime continuum having four coordinates) and the transformation that leaves that distance unchanged. And just like flatlanders, we know that the formula can be dictated only by experimental observation. So, what is that crucial experiment that gives us the invariant distance formula for the 4-dimensional spacetime? It consists of comparing the rate of the ticking of two special identical clocks in relative motion, assuming the independence of the speed of light from the motion of its source (Einstein's second postulate of special relativity).

More specifically, consider a *Michelson-Morley (MM) clock* consisting of a tube of length L with a source (S) of light at one end and a mirror (M) at the other [Figure 1(a)]. The "tick" of the clock is the time that it takes light to go from the source to the mirror and back. For a stationary clock a tick is therefore, $2L/c$.

An MM clock is placed on the train and observed by our two observers O (on the train) and O' (on the ground). Consider the three events constituting a tick: E_1, emission of a light beam at S; E_2, the reflection of light at M; and E_3, its reception at S. Relative to O, the clock is stationary. Therefore, a tick, denoted by $\Delta\tau$, is as above:

$$\Delta\tau = 2\frac{L}{c} \tag{5}$$

How does O' perceive the succession of these three events? Since the clock is moving to the right, the light signal that leaves S will reach M only after M has moved to the right. Thus, to O', the events E_1 and E_2 are separated not only by a vertical distance, but also by a horizontal distance [see Figure 1(b)]. By referring to the right triangle E_1AE_2 of Figure 1 and using the Pythagorean theorem with $\overline{E_1A} = vt$, $\overline{E_2A} = L$, and $\overline{E_1E_2} = ct$, we can show that[1]

$$(ct)^2 = (vt)^2 + L^2 \;\Rightarrow\; t^2 = \frac{L^2/c^2}{1 - v^2/c^2} \;\Rightarrow\; t = \frac{L/c}{\sqrt{1 - v^2/c^2}}$$

Let us denote by Δt the duration of the light's round trip as seen by O'. Then

$$\Delta t = 2t = \frac{2L/c}{\sqrt{1 - v^2/c^2}} = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} \tag{6}$$

where we have used Equation (5).

Although Equation (6) is derived for a single tick, it really applies to all time intervals because any such interval is a multiple of a single tick. If each tick is altered by a factor of $\sqrt{1 - v^2/c^2}$, then a second, an hour, or a year is also altered by the same factor.

What is the difference between Δt and $\Delta\tau$? Both measure the time interval between two events, E_1 and E_3, but $\Delta\tau$ measures the time interval in a RF in which E_1 and E_3 occur at the *same spatial point*: The emission and reception of light occur at the same point S. For this reason $\Delta\tau$ is called **proper time**. We now rewrite Equation (6), realizing that $\Delta\tau$ is the proper time between *any two events*, while Δt is the time measured by a clock relative to which the two events occur at two different spatial points. To be more transparent, let $\Delta\tau = \tau_2 - \tau_1$ be the proper time between two events E_1 and E_2, i.e., $\Delta\tau$ is the time interval as measured by a clock that is present at both events. Let $\Delta t = t_2 - t_1$ be the time interval between the same two events as measured by a clock moving relative to the first clock with speed v. Furthermore, to make lengths out of time intervals it is common to multiply both sides of Equation (6) by c and denote $c\tau$ by s. Then

$$ct_2 - ct_1 = \frac{s_2 - s_1}{\sqrt{1 - v^2/c^2}} \tag{7}$$

[1]We are assuming that L, the length perpendicular to the direction of motion, is not affected by motion (see [5]).

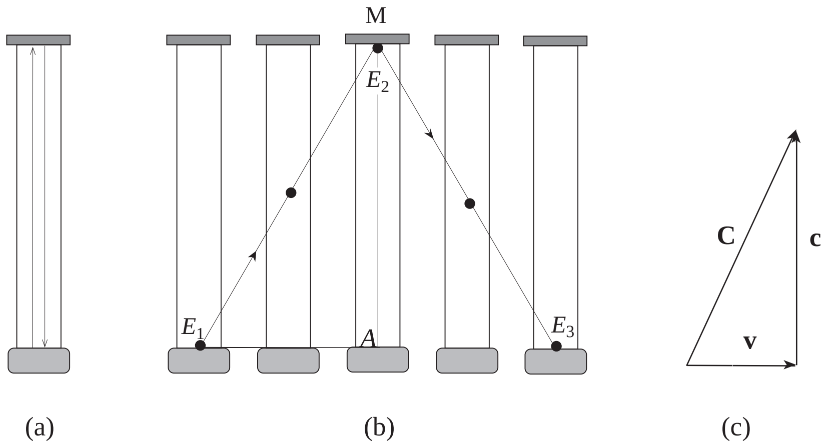

Figure 1: (a) A stationary Michelson–Morley clock. (b) A moving MM clock. The path of light (represented by a black dot) is not a vertical line but a slanted one due to the motion of M. (c) Law of addition of velocities applied to the light signal in the MM clock.

This is the crucial relation that will dictate the formula for the distance in spacetime. Its similarity to Equation (2) is striking. Retracing exactly the same steps following Equation (2), we identify $s_2 - s_1$ as the distance $d(E_1, E_2)$, and note that $v = (x_2 - x_1)/(t_2 - t_1)$. Equation (7) then yields

$$ct_2 - ct_1 = \frac{d(E_1, E_2)}{\sqrt{1 - (x_2 - x_1)^2/(ct_2 - ct_1)^2}}$$

or

$$d(E_1, E_2) = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2} \tag{8}$$

This is the invariant *Minkowskian distance* applicable to the two-dimensional spacetime.

Now that we have the formula for the distance, we must find the transformation that leaves this distance invariant. If observer O assigns coordinates (x_1, ct_1) and (x_2, ct_2) to the two events E_1 and E_2, and observer O' assigns coordinates (x'_1, ct'_1) and (x'_2, ct'_2) to the same two events, how are the primed coordinates related to the unprimed coordinates? We have multiplied the time coordinate by c to make its unit comply with the x-coordinate. Appendix B derives the celebrated Lorentz transformation which leaves the Minkowski distance invariant.

3.1 What would Newton say?

It is instructive to investigate the consequence of abandoning the second postulate of relativity and restoring the law of addition of velocity. Figure 1(c) shows the "new" speed of light C as seen by O'. If this law were true, then O' would measure the speed of light—as it goes from E_1 to E_2—to be C. Then the Pythagorean theorem for the right triangle E_1AE_2 of Figure 1 would become $C^2t^2 = v^2t^2 + L^2$. But Pythagorean theorem applied to Figure 1(c) yields $C^2 = v^2 + c^2$. Therefore,

$$C^2t^2 = v^2t^2 + L^2 \;\Rightarrow\; (v^2 + c^2)t^2 = v^2t^2 + L^2 \;\Rightarrow\; c^2t^2 = L^2$$

or $t = L/c$, leading to

$$\Delta t = 2t = 2L/c = \Delta\tau,$$

i.e., the two observers would measure the same time interval: time would *not* be relative but universal! It should now be clear that the second postulate of relativity is at the heart of the relativity of time.

Appendices

A Space Transformation

Here we want to find a transformation between different coordinate systems which leaves the distance formula (3) unchanged. Suppose that O uses (x, z) for his coordinates and O' uses (x', z') for hers. We are looking for a relation between the primed and unprimed coordinates which respects (3).

Let us start with

$$x' = a + cx + dz, \quad z' = b + ex + fz \tag{9}$$

where a, c, etc. are unknowns to be determined. We have assumed a linear relation (no x^2, z^3, or any other functions), because we want straight lines in O to remain straight lines in O'.

Now take any two points P_1 and P_2 with coordinates (x_1, z_1) and (x_2, z_2) in O and (x'_1, z'_1) and (x'_2, z'_2) in O' and find $\Delta x' \equiv x'_2 - x'_1$ and $\Delta z' \equiv z'_2 - z'_1$ in terms of $\Delta x \equiv x_2 - x_1$ and $\Delta z \equiv z_2 - z_1$:

$$\begin{aligned} \Delta x' &= c(\Delta x) + d(\Delta z) \\ \Delta z' &= e(\Delta x) + f(\Delta z) \end{aligned} \tag{10}$$

Next , demand that the two distances calculated in O and O' be equal. Substituting (10) in $(\Delta x')^2 + (\Delta z')^2 = (\Delta x)^2 + (\Delta z)^2$ and rearranging, we get

$$(c^2 + e^2)(\Delta x)^2 + (d^2 + f^2)(\Delta z)^2 + 2(cd + ef)(\Delta x)(\Delta z) = (\Delta x)^2 + (\Delta z)^2$$

If the two sides of this equation are to be equal for *any* values of Δx and Δz (corresponding to *any* two points in the plane), we must have

$$c^2 + e^2 = 1, \quad d^2 + f^2 = 1, \quad cd + ef = 0 \tag{11}$$

When $\Delta x = 0$, i.e., when P_1 and P_2 lie along the z-axis, we must get $\Delta x'/\Delta z' = m$. Setting $\Delta x = 0$ in Equation (10) gives $m = \Delta x'/\Delta z' = d/f$ or $d = mf$. Substituting this in the second equation of (11) yields $f = \pm 1/\sqrt{1 + m^2}$. With f so determined, we can find d: $d = \pm m/\sqrt{1 + m^2}$. Using the values of d and f (either plus or minus) in the last equation of (11), we get $e = -mc$. Now we can decide which sign to choose. If we choose the negative sign, then Equation (9) would yield $z' = -z$ for $m = 0$, $a = 0$, and $b = 0$ ($e = -mc$ is also zero). But this is impossible, because for these parameters, the two coordinates should coincide. Putting $e = -mc$ in the first equation, we obtain $c = \pm 1/\sqrt{1 + m^2}$. Again, we have to choose the plus sign, because otherwise Equation (9) would yield $x' = -x$ for $m = 0$, $a = 0$, and $b = 0$. Substituting all the unknowns in Equation (9) yields

$$\begin{aligned} x' &= a + \frac{1}{\sqrt{1 + m^2}} x + \frac{m}{\sqrt{1 + m^2}} z \\ z' &= b - \frac{m}{\sqrt{1 + m^2}} x + \frac{1}{\sqrt{1 + m^2}} z \end{aligned} \tag{12}$$

What is the geometric meaning of Equation (12)? Since a and b translate the origin of O' relative to O, we set them equal to zero and assume that the two origins coincide. Let O' draw her coordinates perpendicular to each other as shown in Figure 2(a). How does the z-axis of O look in the $x'z'$-plane? The z-axis is a line; so it must have an equation in terms of x' and z'. In fact, since z is the set of all points whose x are zero, settin $x = 0$ in Equation (12), gives us the equation of the z-axis. With $b = 0$ this gives $z' = x'/m$, which is a line passing through the origin O' and having a slope $1/m$. If we designate the angle between z and z' by α, then

$$\frac{1}{m} = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha = \frac{1}{\tan\alpha} \Rightarrow m = \tan\alpha$$

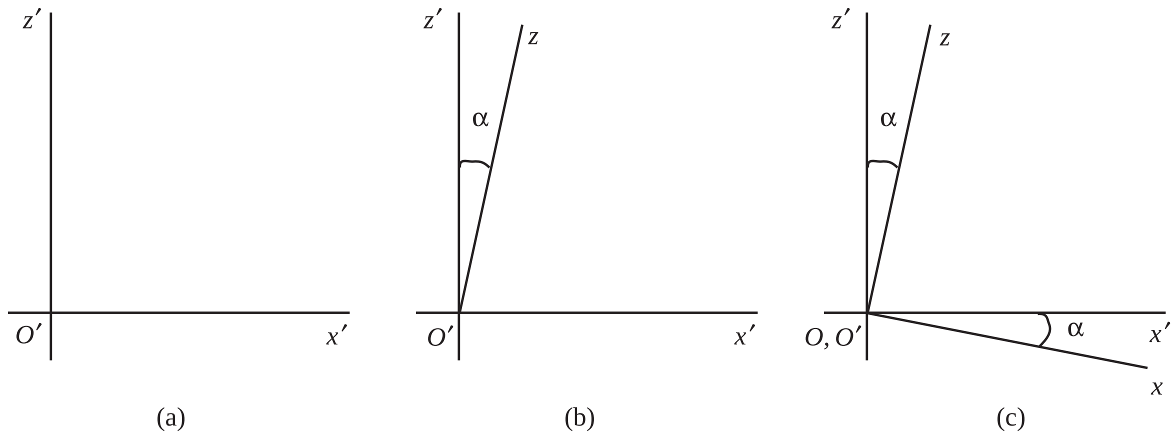

Figure 2: (a) The two perpendicular axes of O'. (b) The z-axis of O makes an angle α with the z'-axis of O'. (c) The axes x and z turn out to be perpendicular to each other.

[see Figure 2(b)]. We can similarly obtain the equation of the x-axis by settin $z = 0$. This yields $z' = -mx'$, showing that the slope of the x-axis is $-m = \tan(-\alpha)$. We conclude that the x-axis makes an angle of $-\alpha$ with the x'-axis, i.e, that *x-axis is perpendicular to the z-axis* [Figure 2(c)].

B Spacetime Transformation

As in Appendix A, we start with

$$x' = a + dx + et, \quad t' = b + fx + gt \tag{13}$$

where a, d, etc. are unknowns to be determined. For the same reason as in Appendix A we are assuming a linear relation. To keep the algebra simple, we write t rather than ct in these equations. We will reintroduce c at the end.

Now take any two events E_1 and E_2, where E_1 has coordinates (x_1, ct_1) in O and (x'_1, ct'_1) in O' and E_2 has coordinates (x_2, ct_2) in O and (x'_2, ct'_2) in O'. These coordinates lead to

$$\begin{aligned} \Delta x' &= d(\Delta x) + e(\Delta t) \\ \Delta t' &= f(\Delta x) + g(\Delta t) \end{aligned} \tag{14}$$

The next step is to demand that $(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2$. Substituting for $\Delta x'$ and $\Delta t'$ in terms of Δx and Δt and rearranging, we get

$$(c^2 f^2 - d^2)(\Delta x)^2 + (c^2 g^2 - e^2)(\Delta t)^2 + 2(c^2 fg - de)(\Delta x)(\Delta t) = c^2(\Delta t)^2 - (\Delta x)^2$$

If the two sides of this equation are to be equal for *any* values of Δx and Δt (corresponding to *any* two events in the spacetime plane), we must have

$$c^2 f^2 - d^2 = -1, \quad c^2 g^2 - e^2 = c^2, \quad c^2 fg - de = 0 \tag{15}$$

When $\Delta x = 0$, i.e., when E_1 and E_2 occur at the same point along x-axis, we must get $\Delta x'/\Delta t' = v$ the speed of that point on the x-axis relative to O'. Setting $\Delta x = 0$ in Equation (14) gives $v = \Delta x'/\Delta t' = e/g$ or $e = gv$. Furthermore, since E_1 and E_2 occur at the same point, Δt is the proper time. It follows from Equation (6) and the second equation in (14) that $g = 1/\sqrt{1-(v/c)^2}$.

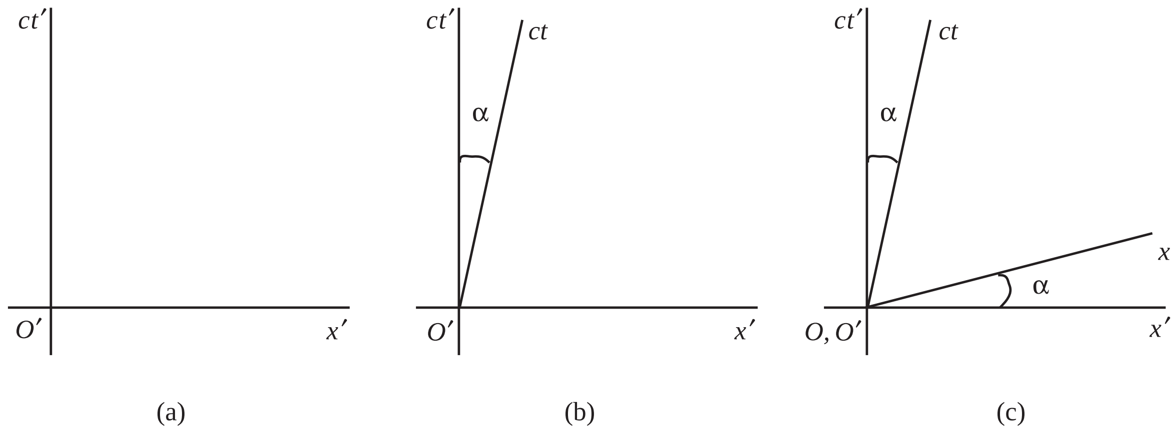

Figure 3: (a) The two perpendicular axes of O'. (b) The ct-axis of O makes an angle α with the ct'-axis of O'. (c) The axes x and ct are *not* perpendicular to each other.

With e and g so determined, the last equation in (15) yields $f = dv/c^2$. Substituting this in the first equation of (15), we get $d = \pm 1/\sqrt{1-(v/c)^2}$. Only the positive sign is acceptable, because when $a = 0$, $b = 0$, and $v = 0$, x' must equal x (the two RFs are not moving relative to one another).

Using the familiar symbol $\gamma = 1/\sqrt{1-(v/c)^2}$, Equation (13) can be written as

$$x' = a + \gamma x + \gamma v t, \quad t' = b + \gamma(v/c^2)x + \gamma t$$

Specializing to $a = 0$ and $b = 0$ and introducing the new and commonly used notation $\beta = v/c$ (or $v = \beta c$), we can write the first equation as

$$x' = \gamma x + \gamma\beta ct = \gamma(x + \beta ct)$$

Multiply both sides of the second equation by c to get

$$ct' = c\gamma(v/c^2)x + c\gamma t = \gamma(v/c)x + \gamma ct = \gamma(\beta x + ct)$$

Putting everything together, we obtain the celebrated **Lorentz transformation**:

$$\begin{aligned} x' &= \gamma(x + \beta ct) \\ ct' &= \gamma(\beta x + ct) \end{aligned} \tag{16}$$

What is the geometric meaning of Equation (16)? Let O' draw her coordinates perpendicular to each other as shown in Figure 3(a). How does the ct-axis of O look in the $x'ct'$-plane? Since ct is the set of all points whose x are zero, setting $x = 0$ in Equation (16), gives us the equation of the ct-axis: $ct' = x'/\beta$, which is a line passing through the origin O' and having a slope $1/\beta$. If we designate the angle between ct and ct' by α, then

$$\frac{1}{\beta} = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha = \frac{1}{\tan\alpha} \;\Rightarrow\; \beta = \tan\alpha$$

[see Figure 3(b)]. We can similarly obtain the equation of the x-axis by setting $t = 0$. This yields $ct' = \beta x'$, showing that the slope of the x-axis is $\beta = \tan(\alpha)$. We conclude that the x-axis makes an angle of $\pi/2 - 2\alpha$ with the x'-axis, i.e, that *x-axis is not perpendicular to the ct-axis* [Figure 3(c)].

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