Santa and Conservation of Energy

1 Chicago-New York Fuel

Assuming that Santa's "vehicle" is 1000 kg, the energy needed to speed it up to 65 mph (or about 30 m/s) is just the final kinetic energy, which is

$$KE = \tfrac{1}{2}(1000)(30)^2 = 450,000 \text{ J}.$$

Now, each kilogram of gasoline contains around 4 million Joules. So, Santa needs 112.5 grams of gasoline. The density of gasoline is about 0.75 kg/m^3. This puts the volume of the fuel consumed in Santa's trip from Chicago to NY at 150 cc or about five ounces. So, a few "drops" is a *little* exaggeration! A few ounces is more accurate.

2 Chimney Energy

We want to find the time that Santa has to spend in a typical chimney for his energy consumption to be minimum. We let t stand for the time going down or coming up a chimney. Eventually we calculate t in such a way as to minimize the total energy consumption per house. There are two kinetic energies involved: One for chimney plunging and climbing, the other for hopping from one chimney to the next. For his trip down the chimney, Santa travels 4 m carrying a mass of 120 kg (himself plus 10 two-kg toys for 10 children). His speed is therefore $4/t$. It follows that the KE of descent is

$$KE_{\text{down}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(120)(4/t)^2 = \frac{960}{t^2}.$$

When Santa climbs up the chimney, he is a little lighter (20 kg lighter for leaving the toys behind). Therefore, his KE of ascent is

$$KE_{\text{up}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(100)(4/t)^2 = \frac{800}{t^2}.$$

and the total KE for his "chimney travel" is

$$KE_{\text{chim}} = KE_{\text{down}} + KE_{\text{up}} = \frac{960}{t^2} + \frac{800}{t^2} = \frac{1760}{t^2}.$$

3 Hopping Energy

The hopping energy is a little more complicated. Santa starts out with a lot of toys; but as he delivers the toys, their number decreases. Intuitively, we may want to take the *effective* mass of Santa plus the toys to be the average of the initial and final masses (the latter being zero because Santa's mass is negligible compared with the total mass of the toys) or half the mass he starts with. It turns out that an exact calculation of the hopping kinetic energy leads to the same result (see the Appendix).

To find the hopping KE, we note (from video) that the entire time available to Santa for going from one house to the next is 0.00576 second. Out of this, $2t$ seconds is spent plunging down and climbing up the chimney. So, $5.76 \times 10^{-3} - 2t$ is left to cover the distance of 10 m between two adjacent chimneys. Therefore, the hopping kinetic energy is

$$\begin{aligned} KE_{\text{hop}} &= \tfrac{1}{2} m_{\text{eff}} v^2 = \tfrac{1}{2}(1.5 \times 10^8)\left(\frac{10}{5.76 \times 10^{-3} - 2t}\right)^2 \\ &= (7.5 \times 10^7)\left[\frac{100}{(5.76 \times 10^{-3} - 2t)^2}\right] = \frac{7.5 \times 10^9}{(5.76 \times 10^{-3} - 2t)^2} \end{aligned} \tag{1}$$

4 Minimizing Energy

The total energy *as a function of* t is

$$KE(t) = KE_{\text{chim}} + KE_{\text{hop}} = \frac{1760}{t^2} + \frac{7.5 \times 10^9}{(5.76 \times 10^{-3} - 2t)^2}.$$

What value of t minimizes this? You can either use calculus to answer this question through the process of differentiation, or you can simply plot $KE(t)$ as a function of t and see where the minimum occurs. Figure 1 shows $KE(t)$ as a function of t for points close to the minimum. It is clear that this minimum occurs at almost 0.000028 or 28 microseconds. So, Santa spends 56 microseconds per chimney; and if the chimney is 4 meters long, its speed would be

$$\frac{4}{0.000028} = 142857 \text{ m/s} = 142.857 \text{ km/s}.$$

To accelerate to this speed on his way down (carrying himself and 20 kg of toys), Santa needs

$$KE_{\text{down}} = \tfrac{1}{2}(120)(142857)^2 = 1.225 \times 10^{12} \text{ J}.$$

On his way up, he needs

$$KE_{\text{up}} = \tfrac{1}{2}(100)(142857)^2 = 1.02 \times 10^{12} \text{ J}.$$

For his hopping energy, use Equation (1) and substitute 28 microseconds for t. This gives

$$KE_{\text{hop}} = \frac{7.5 \times 10^9}{(5.76 \times 10^{-3} - 0.000056)^2} = 2.38 \times 10^{14} \text{ J}.$$

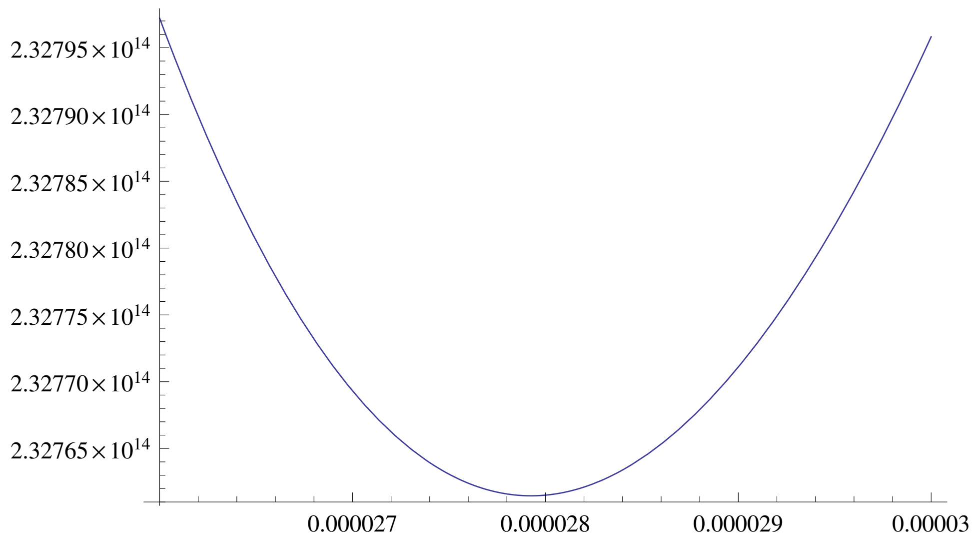

Figure 1: The total KE that Santa consumes per house as a function of time.

So, the total energy Santa uses *per house* is

$$1.225 \times 10^{12} + 1.02 \times 10^{12} + 2.38 \times 10^{14} = 2.4 \times 10^{14} \text{ J.}$$

And this is the minimum amount of energy physically possible. The graph also shows the minimum KE, which appears to be about 2.3275×10^{14} Joules. The slight discrepancy is due to our rounding the time up to 28 microsecond. It is actually a little less than that.

5 Total Energy Consumption

Since there are 15 million houses, the total energy consumption by Santa is

$$15,000,000 \times 2.4 \times 10^{14} = 3.6 \times 10^{21}.$$

How does this compare with the yearly energy consumption of the world? The annual energy consumption of the entire world in 2010 was about 500 quadrillion Btu (a Btu is 1050 Joules),[1] or about

$$500 \times 10^{15} \times 1050 = 5.25 \times 10^{20} \text{ J.}$$

This includes not only the typical residential usage such as heating, lighting, cooking, commuting, and entertainment, but also the large scale industrial, agricultural, and transportation consumption. Santa uses $3.6 \times 10^{21}/5.25 \times 10^{20} = 6.86$ times the annual world energy supply—of any form—*in one day*!

[1] See www.eia.gov/forecasts/ieo/

6 Explosions

We found that Santa has a kinetic energy of 1.225×10^{12} Joules when he reaches the bottom of the chimney. This energy turns into heat when he brakes to a complete stop. For comparison, the heat produced in the explosion of a ton of TNT is about 4×10^9 Joules. His KE as he lands at the next chimney was calculated to be 2.38×10^{14} Joules, and this energy is turned into heat in a few milliseconds, qualifying this release of energy as an explosion as well. How many tons of TNT is this equivalent to? $2.38 \times 10^{14}/4 \times 10^9 = 59500$ tons of TNT! The destructive power of "Little Boy," the bomb that was dropped on Hiroshima, was a "mere" 15,000 tons of TNT and that of the "Fat Man," the bomb dropped on Nagasaki was 22000 tons of TNT. Every time Santa lands at a chimney, he detonates almost twice the two bombs combined, and he is at ground zero of every blast!

7 Appendix

Instead of using numbers, let's use symbols. That way, we don't have to keep writing powers of ten. Let N_c be the number of children and n the number of children per house, so that the number of houses, N_h is $N_h = N_c/n$. Let M denote Santa's mass and m, the mass of each toy. Thus, the hopping KE for the first house is

$$KE_1 = \tfrac{1}{2}(N_c m + M)v^2.$$

As he leaves n toys in the first house, his mass decreases by mn. Therefore, the hopping KE for the second house is

$$KE_2 = \tfrac{1}{2}[N_c m - nm + M]v^2 = \tfrac{1}{2}(N_c m + M)v^2 - \tfrac{1}{2}nmv^2.$$

For the third house, it is

$$KE_3 = \tfrac{1}{2}[N_c m - nm - nm + M]v^2 = \tfrac{1}{2}(N_c m + M)v^2 - 2 \times \tfrac{1}{2}nmv^2,$$

and for the kth house it is

$$KE_k = \tfrac{1}{2}[N_c m - (k-1)nm + M]v^2 = \tfrac{1}{2}(N_c m + M)v^2 - (k-1) \times \tfrac{1}{2}nmv^2,$$

and for the last house it is

$$\begin{aligned} KE_{N_h} &= \tfrac{1}{2}[N_c m - (N_h - 1)nm + M]v^2 \\ &= \tfrac{1}{2}(N_c m + M)v^2 - (N_h - 1) \times \tfrac{1}{2}nmv^2. \end{aligned}$$

Adding all these, we get a total KE of

$$\begin{aligned} KE =& N_h[\tfrac{1}{2}(N_c m + M)v^2] - \tfrac{1}{2}nmv^2\,(1 + 2 + 3 + \cdots + N_h - 1) \\ =& N_h\left[\tfrac{1}{2}(N_h nm + M)v^2\right] - \tfrac{1}{2}nmv^2\left[\frac{N_h(N_h-1)}{2}\right] \\ =& \tfrac{1}{4}N_h^2 nmv^2 + \tfrac{1}{2}\left(N_h M + \frac{mN_h n}{2}\right)v^2 \end{aligned}$$

Dividing this by the number of houses N_h, we get

$$KE_{\text{hop}} = \tfrac{1}{4} n N_h m v^2 + \tfrac{1}{2}\left(M + \frac{mn}{2}\right) v^2 \approx \tfrac{1}{2}\left(\frac{N_c m}{2}\right) v^2$$

because $nN_h m = N_c m$ is much greater than M and $mn/2$. This shows that the effective mass is $N_c m/2$.